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Technical Note

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Error Performance Bounds  
for Two Receivers  
for Optical Communication  
and Detection

E. A. Bucher

10 November 1970

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ERROR PERFORMANCE BOUNDS FOR TWO RECEIVERS  
FOR OPTICAL COMMUNICATION AND DETECTION

*E. A. BUCHER*

*Group 65*

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## ABSTRACT

This report examines two possible receiver strategies for use with photoelectron emitting optical detectors. The Poisson statistics of these photoelectron emissions are used to find simple easily evaluated but tight upper bounds on error probability with both receiver decision rules. Upper bounds on error probability are derived for both M-ary PPM communication with a maximum likelihood receiver and for a fixed threshold radar detection receiver.

These receiver performance bounds illustrate several differences between optical or quantum communication and conventional communication. These differences are discussed in detail.

Accepted for the Air Force  
Joseph R. Waterman, Lt. Col., USAF  
Chief, Lincoln Laboratory Project Office



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ERROR PERFORMANCE BOUNDS  
FOR TWO RECEIVERS  
FOR OPTICAL COMMUNICATION AND DETECTION

## I. INTRODUCTION

A number of authors<sup>1-6</sup> have considered optical communication and detection receiver strategies. The error probabilities for these receivers have been expressed either in terms of infinite summations which have no closed form solution or in terms of sums involving Bessel functions, etc., which are not readily calculated. Some authors have presented graphs of error probability for selected values of system parameters. The calculate and plot approach does not readily lead to much insight in the tradeoffs between various design parameters and certainly does not provide a quick way of evaluating the effects of changing design parameters.

This report presents some relatively simple but rather tight analytic upper bounds on error probability for both optical communication and detection. These bounds are then used to demonstrate some of the basic characteristics of optical detection and communication.

Before presenting these bounds, we review the basic properties of optical receivers. Next, we consider the error bound for optical communication; then the bounds for optical detection. Finally, we discuss these bounds and their implications.

## II. DIRECT-DETECTION OPTICAL RECEIVERS

In optical receivers, the basic physical process in the detector is the emission of photoelectrons from a photosensitive surface. Since the optical detector produces these emissions, the most reasonable way to define the receiver is in terms of these emissions. Photoelectron emissions are discrete independent random events subject to the laws of quantum physics. These emissions are usually modeled as occurring with Poisson statistics of mean event rate, or average occurrence rate,  $\lambda$ . The parameter  $\lambda$  is a function of the entire optical system and is calculated as

$$\lambda = \frac{\eta W_i}{h\nu} \quad , \quad (1)$$

where

$W_i$  = Power incident on the detector

$\eta$  = Detector quantum efficiency

$h\nu$  = Energy in a quantum of light of frequency  $\nu$ .

With Poisson statistics, an average of  $\lambda\tau$  events (emissions) occur in time interval  $\tau$ ; furthermore, the probability of exactly  $n$  events in time interval  $\tau$ , given the mean event rate  $\lambda$ , is

$$P(n/\lambda) = \frac{(\lambda\tau)^n}{n!} e^{-\lambda\tau} \quad . \quad (2)$$

Another basic property of the Poisson process is that the number of events occurring in disjoint time intervals are statistically independent. For such photoemissive or "direct" detectors, all signaling must be accomplished by changing the  $W_i$  incident on the detector. Finally, all photo-detectors have some inherent dark current which must be included in calculating the  $\lambda$  due to

nonsignal or background radiation. For the rest of this report, we shall assume that the mean event rate due to all background or nonsignal power is  $B$ , and that the addition of signal power increases this event rate to  $S + B$ .

### III. COMPARISON RECEIVER FOR COMMUNICATION

A commonly envisioned optical modulation system used for photoemissive detectors is pulse-position modulation or digital PPM. If one of  $M$  symbols,  $m_1$  through  $m_M$ , is to be communicated, the PPM system signals by transmitting signal energy in the one of  $M$  possible time slots assigned to the desired symbol and no signal in the  $M-1$  time slots corresponding to the other  $M-1$  symbols. The transmitter is assumed to operate at full intensity during the entire transmitting interval. Each time slot is assumed to be  $\tau$  seconds wide. It is also assumed that no signal energy is present in any of the  $M-1$  time intervals corresponding to the other  $M-1$  symbols. If any cross talk is present, the cross talk must be considered as increasing the background rate  $B$  observed during those nonsignal intervals.

The PPM system is most easily analyzed for the simple case  $M = 2$ . A signal is transmitted in time slot 1 to communicate  $m_1$  and in time slot 2 to communicate  $m_2$ . For constant intensity signals, all the signal dependent information at the receiver is contained in the number of photoelectron emissions,  $n_1$  and  $n_2$ , counted during receiver time slots 1 and 2, respectively. If  $m_1$  and  $m_2$  occur with equal probability, the maximum likelihood receiver is optimum, and the decision rule is to select the  $m$  which maximizes the conditional probability of  $n_1$  and  $n_2$  given  $m$ . This decision rule may be symbolically written as

$$P(n_1, n_2/m_1) \underset{m_2}{\overset{m_1}{\geq}} P(n_1, n_2/m_2) \quad . \quad (3)$$

Using Eq. (2) and the independence of  $n_1$  and  $n_2$ , we may evaluate the decision rule as

$$\frac{[(S+B)\tau]^{n_1}}{n_1!} e^{-(S+B)\tau} \underset{m_2}{\overset{m_1}{\geq}} \frac{[B\tau]^{n_2}}{n_2!} e^{-B\tau} \frac{[B\tau]^{n_1}}{n_1!} e^{-B\tau} \frac{[(S+B)\tau]^{n_2}}{n_2!} e^{-(S+B)\tau} \quad .$$

Some algebra simplifies the decision rule to

$$\left(\frac{S+B}{B}\right)^{n_1} \underset{m_2}{\overset{m_1}{\geq}} \left(\frac{S+B}{B}\right)^{n_2} \quad .$$

Since  $(S+B)/B$  is greater than 1, the sense of the inequalities is preserved by taking logarithms on both sides, hence the  $M = 2$ , PPM decision rule is

$$n_1 \underset{m_2}{\overset{m_1}{\geq}} n_2 \quad .$$

If  $m_1$  is transmitted, a communication error can occur only if  $n_1 < n_2$  and vice versa. By symmetry, the error probability is the same whether  $m_1$  or  $m_2$  was transmitted. Thus

$$P(E) = P[n_1 < n_2/m_1] \quad . \quad (4)$$



Explicit evaluation of  $P(E)$  involves a sum of terms involving Bessel functions.<sup>2,3</sup> A simple but rather tight upper bound on  $P(E)$  may be obtained by using the Chernoff bound.<sup>7</sup> In Appendix A, we find a bound on  $P(E)$  which shows that

$$P(E) \leq e^{-S\tau E_C(S/B)} \quad (5)$$

for binary PPM where

$$E_C\left(\frac{S}{B}\right) = \left[1 - 2 \frac{B}{S} \left(\sqrt{1 + \frac{S}{B}} - 1\right)\right] = \left[\frac{[\sqrt{1 + (S/B)} - 1]^2}{S/B}\right] \quad (6)$$

The error exponent  $E_C(S/B)$  is only a function of the ratio  $S/B$  and not a function of  $\tau$ .

For full  $M$ -ary PPM, the receiver decision rule is to select the symbol corresponding to the receiver time interval with the largest number of counted photoelectron emissions. By symmetry the error probability is the same for each possible transmitted symbol. Thus

$$\begin{aligned} P(E) &= P(E/m_1 \text{ transmitted}) \\ &= P[n_1 \leq \max(n_2, \dots, n_M)/m_1] \end{aligned} \quad (7)$$

The union bound allows us to upper bound the right side of Eq. (7) as

$$P(E) \leq \sum_{j=2}^M P[n_1 \leq n_j/m_1] \quad (8)$$

Since  $n_2$  through  $n_M$  are statistically independent, identically distributed random variables given that  $m_1$  is transmitted

$$P[n_1 \leq n_2/m_1] = P[n_1 \leq n_j/m_1]$$

for all  $j$  between 2 and  $M$ . Thus,

$$P(E) \leq (M-1) P[n_1 \leq n_2/m_1] \quad (9)$$

But Eqs. (5) and (6) upper bound  $P[n_1 \leq n_2/m_1]$ , thus

$$P(E) \leq (M-1) e^{-S\tau E_C(S/B)} \quad (10)$$

for  $M$ -ary PPM.

#### IV. THRESHOLD RECEIVER FOR DETECTION AND COMMUNICATION

In the detection problem the receiver must decide whether there is signal present, not which signal is present. For the case in point, the detection receiver must decide whether there is signal power present along with background power by observing for a time interval  $\tau$ . Thus the receiver determines whether  $\lambda = S + B$  or  $\lambda = B$  by looking at the process for time  $\tau$ . Let hypothesis  $H_0$  be that background alone is present ( $\lambda = B$ ) and  $H_1$  be the hypothesis that both signal and background are present ( $\lambda = S + B$ ).

The exact method of formulating the detection receiver decision rule differs slightly, depending upon whether one uses Bayes rule<sup>8</sup> to minimize the risk given a set of costs or whether one uses a Neyman-Pearson test<sup>8</sup> to obtain the best performance consistent with a specified probability of saying  $H_1$  given that  $H_0$  is true. Let us assume that the receiver has observed  $n$  counts.

Both the Bayes and Neyman-Pearson tests are likelihood ratio tests<sup>8</sup> in which the likelihood ratio

$$\frac{P(n/H_1)}{P(n/H_0)}$$

is compared with a threshold  $\Gamma$  and the receiver guesses that  $H_1$  is true if the likelihood ratio exceeds  $\Gamma$  and that  $H_0$  is true otherwise. This detection rule is written as

$$\frac{P(n/H_1)}{P(n/H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \Gamma \quad . \quad (11)$$

Thus the decision rule is

$$\frac{P(n/\lambda = S+B)}{P(n/\lambda = B)} \underset{H_0}{\overset{H_1}{\gtrless}} \Gamma \quad .$$

Using Eq. (2) for  $P(n/\lambda)$ , we find that the test becomes

$$\frac{\left[ \frac{(S+B)\tau}{n!} e^{-(S+B)\tau} \right]^n}{\frac{(B\tau)^n}{n!} e^{-B\tau}} \underset{H_0}{\overset{H_1}{\gtrless}} \Gamma \quad . \quad (12)$$

Taking natural logarithms of both sides of the inequality and performing some algebra we find that the decision rule is

$$n \underset{H_0}{\overset{H_1}{\gtrless}} T \quad (13)$$

where

$$T = \frac{S\tau + \ln \Gamma}{\ln(1 + S/B)} \quad . \quad (14)$$

The quantity  $T$  is often called the detector threshold.

There are two basic errors which may occur. The receiver may estimate  $H_1$  when  $H_0$  is true or  $H_0$  when  $H_1$  is true. Conventionally these two events are called a "false alarm" and a "miss" respectively and occur with probabilities  $P_F$  and  $P_M$  respectively.

A good but simple bound on  $P_F$  and  $P_M$  provides a way of estimating the implications of various values of  $S$ ,  $B$  and  $\tau$  without performing tedious calculations. As above, the Chernoff<sup>7</sup> bound provides such a bound. In Appendix B, it is shown that

$$P_M < e^{-S\tau[1 + \frac{B}{S}]} \left[ \frac{\alpha - \ln \alpha - 1}{\alpha} \right] \quad (15)$$

$$P_F < \frac{1}{\Gamma} e^{-S\tau[1 + \frac{B}{S}]} \left[ \frac{\alpha - \ln \alpha - 1}{\alpha} \right] \quad (16)$$

where

$$\alpha = \frac{(S+B)\tau}{T} = \frac{(S+B)\tau \ln(1 + \frac{S}{B})}{S\tau + \ln(\Gamma)} \quad . \quad (17)$$

For the general detection problem with arbitrary  $\Gamma$ , no additional simplifications are possible for the bound in Eqs. (15) to (17). An important special case occurs for  $\Gamma = 1$  which is used for binary communications decisions with equally likely hypotheses. For this special case,

$$\alpha = (1 + \frac{B}{S}) \ln(1 + \frac{S}{B}) \quad (18)$$

and

$$P_M = P_F = P(E) \leq e^{-S\tau E_T(S/B)} \quad (19)$$

where

$$E_T(\frac{S}{B}) = \frac{(1 + \frac{B}{S}) \ln(1 + \frac{S}{B}) - \ln[(1 + \frac{B}{S}) \ln(1 + \frac{S}{B})] - 1}{\ln[1 + \frac{S}{B}]} \quad (20)$$

Figure 1 shows plots of  $E_C(S/B)$  and  $E_T(S/B)$ . The subscripts C and T were chosen to indicate the comparison and threshold receivers, respectively. An intuitive explanation of the

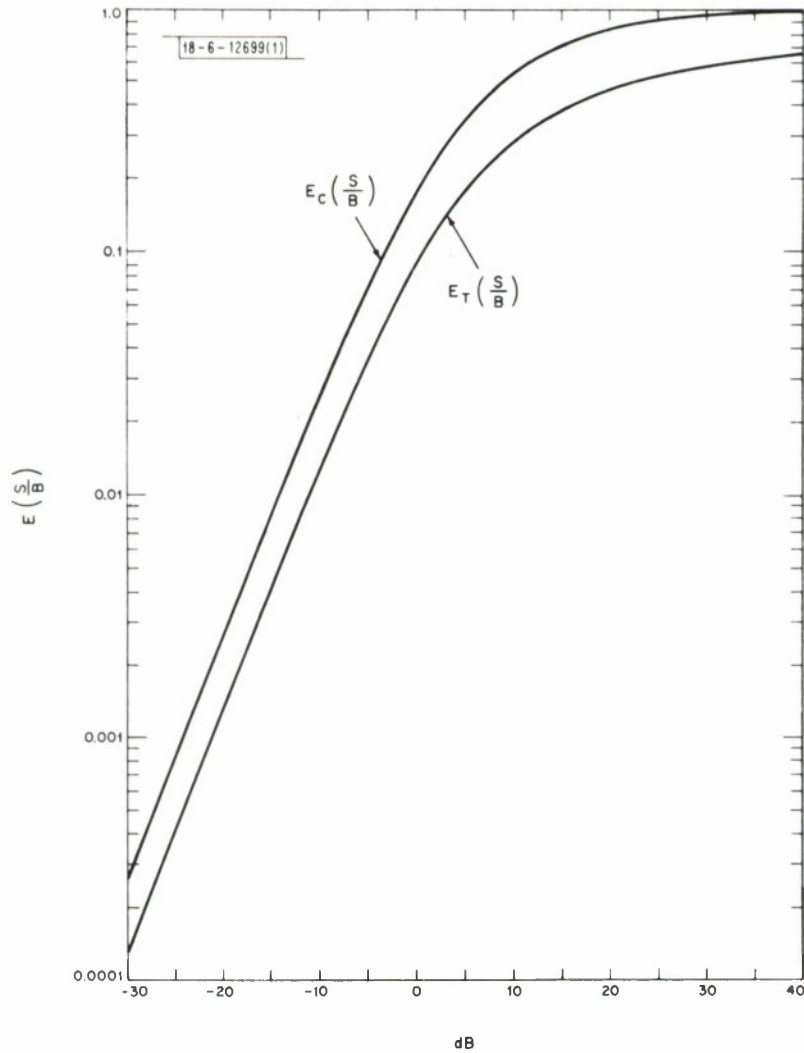


Fig. 1.  $E_C(S/B)$  and  $E_T(S/B)$  as a function of the ratio  $S/B$ .

better performance of the comparison receiver is that the comparison receiver adjusts its "threshold" to just barely exceed the actual background level at each interval, whereas, the threshold or detection receiver must set a fixed threshold high enough to exceed almost all the possible background count numbers. In some sense, we may argue that the comparison receiver has the simpler job of deciding which of two bins contains signal while the detection threshold receiver must also consider whether there is any signal present at all.

## V. DISCUSSION

The asymptotic behavior of the error probability expressions above illustrates two basic differences between optical communication and conventional communication techniques. First let us consider very high signal-to-background ratios. In particular, let us approach the limit of zero background. To evaluate the error probability expressions in Eqs. (5), (10) and (19), we need to know the behavior of  $E_C(S/B)$  and  $E_T(S/B)$  as  $S/B$  goes to infinity. By examining Fig. 1 or performing some algebra we find that

$$\frac{S}{B} \lim_{\rightarrow \infty} E_C\left(\frac{S}{B}\right) = 1 \quad (21)$$

and that

$$\frac{S}{B} \lim_{\rightarrow \infty} E_T\left(\frac{S}{B}\right) = 1 \quad (22)$$

and that these limits are approached from below. Thus even in the total absence of background radiation  $P(E)$  is not zero but

$$P(E) \leq e^{-S\tau} \quad (23)$$

This behavior occurs not because of the error probability bounding but because of the basic quantum mechanical nature of optical detectors; that is, the right side of inequality (23) is identical to the probability that the optical detector will emit no photoelectrons in time  $\tau$  even though it receives a signal strong enough to produce an average of  $S\tau$  photoelectrons in time  $\tau$ .

Now let us consider the case in which the background exceeds the signal by at least a factor of three. In this region

$$\text{and } \left. \begin{aligned} E_C\left(\frac{S}{B}\right) &\approx \frac{S}{4B} \\ E_T\left(\frac{S}{B}\right) &\approx \frac{S}{8B} \end{aligned} \right\} \frac{S}{B} < 0.3$$

Thus Eqs. (5) and (19) become

$$P(E) \leq e^{-S\tau(S/4B)}$$

for the comparison receiver at low  $S/B$  and

$$P(E) \leq e^{-S\tau(S/8B)}$$

for the threshold detection receiver for low  $S/B$ . As an example let us select  $S/B = 0.1$  and  $S\tau = 500$ ; with either receiver structure, this combination of parameters leads to reasonably small error probabilities despite a signal-to-background ratio of  $-10$  dB. This performance

occurs because it is not necessary for the signal to overwhelm the background but only to produce a change which is perceptible above the fluctuations inherent in the background. For the set of numbers used in the simple example above, an average of  $B\tau = 5000$  background photoelectrons will be emitted during the signaling interval but the standard deviation in this number of background photoelectrons is only 71 and an additional 500 signal photoelectrons are usually observable.

Some interesting results can be obtained if we multiply both the numerator and denominator of the argument  $S/B$  used in the  $E(\cdot)$  functions by  $\tau$ . Thus Eqs. (5) and (19) become

$$P(E) \leq e^{-S\tau E_C(S\tau/B\tau)} \quad (24)$$

for the binary-comparison receiver in PPM and

$$P(E) \leq e^{-S\tau E_T(S\tau/B\tau)} \quad (25)$$

for the threshold receiver. But  $S\tau$  equals  $N_S$ , the average number of photoelectrons due to signal, and  $B\tau$  equals  $N_B$ , the average number of photoelectrons due to noise. Rewriting Eqs. (24) and (25) we find that

$$P(E) \leq e^{-N_S E_C(N_S/N_B)}$$

for the binary-comparison receiver in PPM and

$$P(E) \leq e^{-N_S E_T(N_S/N_B)}$$

for the threshold detection receiver.

At this point, much could be said about using sophisticated and expensive coding techniques, like sequential decoding,<sup>9</sup> to achieve reliable communication despite large values of  $P(E)$ ; however, such techniques are difficult to implement and are best used only as a last resort. An often adequate and cheaper system design technique is to design for a reasonably good error probability and then use a simple coding technique, such as threshold decoding,<sup>10</sup> to protect against occasional errors. For such a "good" channel  $P(E) \approx 10^{-4} \approx e^{-10}$  is a conservatively reasonable figure. We can be sure that such performance is possible if the exponent on the right side of the  $P(E)$  inequalities is less than  $-10$ . Thus one can expect good communication performance whenever

$$10 \leq N_S E_C\left(\frac{N_S}{N_B}\right) \quad (26)$$

for the binary PPM receiver and

$$10 \leq N_S E_T\left(\frac{N_S}{N_B}\right) \quad (27)$$

for the threshold detection receiver. As pointed out above,  $E_C(\cdot)$  and  $E_T(\cdot)$  never exceed 1, thus a minimum of 10 photoelectrons is essential for reliable communication even in the complete absence of background. With some albeit small background, intensity 15 to 40 dB below the signal, we may use Fig. 1 to estimate the minimum value of  $N_S$  necessary to satisfy inequalities (26) and (27). In this range something like an average of 10 to 14 signal photoelectrons are



necessary for reliable communication with the binary PPM comparison receiver and an average of 15 to 25 signal photoelectrons are necessary for reliable communication with the binary threshold detection receiver.

We may also estimate possible communication rates  $R$  possible with these two optical modulation schemes by using the expressions for  $P(E)$  to determine the minimum  $\tau$  consistent with reliable communication. If little time is lost in switching the transmitter on and off, the PPM system transmits  $\log_2 M$  bits in  $M\tau$  seconds. Thus the data rate

$$R = \frac{\log_2 M}{M\tau} \text{ bits/sec} \quad . \quad (28)$$

If we require a probability of symbol error less than or equal to  $P_d$ , a sufficient condition is that the upper bound be less than or equal to  $P_d$ . Thus

$$\ln P(E) \leq \ln(M-1) - S\tau E_e\left(\frac{S}{B}\right) \leq \ln P_d$$

or

$$\frac{1}{\tau} \leq \frac{SE_e\left(\frac{S}{B}\right)}{\ln\left[\frac{M-1}{P_d}\right]} \quad . \quad (29)$$

Substituting inequality (29) into Eq. (28) we find that

$$R \leq \frac{\log_2(M)}{M} \frac{SE_e\left(\frac{S}{B}\right)}{\ln\left[\frac{M-1}{P_d}\right]} \quad (30)$$

for M-ary PPM. A very similar argument estimates

$$R \leq \frac{SE_T\left(\frac{S}{B}\right)}{\ln\left(\frac{1}{P_d}\right)} \quad (31)$$

for the threshold detection receiver used with binary on-off keying to transmit one bit every  $\tau$  seconds.

Looking at inequalities (30) and (31) and the asymptotic behavior of both  $E_e(S/B)$  and  $E_T(S/B)$  we note that for large  $S/B$  the possible data rate is essentially proportional to  $S$  in that the  $E(\quad)$  functions are nearly constant for large  $S/B$ . On the other hand, for  $(S/B) < 0.3$ , both  $E(\quad)$  functions are proportional to  $S/B$ . Thus

$$R \propto \frac{S^2}{B}$$

for small  $S/B$ .

Finally, we may use inequalities (30) and (31) to investigate the effects of a pulse transmitter which operates with an additional power gain  $G$  at duty cycle  $1/G$ . For this pulsing transmitter, inequality (30) becomes

$$R \leq \frac{1}{G} \frac{\log_2(M)}{M} \frac{SGE_e\left(\frac{SG}{B}\right)}{\ln\left(\frac{M-1}{P_d}\right)} \quad (32)$$

$$R \leq \frac{\log_2(M)}{M} \frac{SE_c\left(\frac{SG}{B}\right)}{\ln\left(\frac{M-1}{P_d}\right)} \quad (33)$$

Similarly inequality (31) becomes

$$R \leq \frac{SE_T\left(\frac{SG}{B}\right)}{\ln\left(\frac{1}{P_d}\right)} \quad (34)$$

Thus the only change in  $R$  for a pulsing transmitter is replacing  $S/B$  in the  $E(S/B)$  functions by  $SG/B$ . Since the  $E(S/B)$  functions increase linearly with  $S/B$  for  $(S/B) < 0.3$ , we see that the data rate increases by  $G$  for a pulse transmitter if  $(SG/B) < 0.3$ . On the other hand, the  $E(S/B)$  functions are more or less constant for  $(S/B) > 10$ ; thus little gain results from a pulsing transmitter when  $(S/B) > 10$ .

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APPENDIX A  
UPPER BOUND ON P(E) FOR OPTICAL PPM COMPARISON RECEIVER

We wish to upper bound

$$P(E) = P[n_1 < n_2/m_1] \quad (A-1)$$

for the case where  $n_1$  and  $n_2$  are Poisson random variables and the mean of  $n_1$  is  $(B + S) \tau$  and the mean of  $n_2$  is  $B\tau$ . Consider a function  $u(n_1, n_2)$  defined as

$$u(n_1, n_2) = \begin{cases} 1 & \text{if } n_1 < n_2 \\ 0 & \text{if } n_2 < n_1 \end{cases}.$$

Then

$$P(E) = \overline{u(n_1, n_2)} \quad (A-2)$$

where the overbar denotes expectation over all values of  $n_1$  and  $n_2$ . This expectation is just the complicated expression discussed in the text. However, we may upper bound  $P(E)$  by upper bounding  $u(n_1, n_2)$ . For any real number  $\gamma$  greater than or equal to one,

$$u(n_1, n_2) \leq \gamma^{n_2 - n_1} \quad (A-3)$$

because  $\gamma$  to any power is never negative and  $\gamma^{n_2 - n_1}$  is always greater than or equal to one whenever  $n_1 \leq n_2$ . Substituting (A-3) into (A-2) we find that

$$P(E) \leq \overline{\gamma^{n_2 - n_1}}. \quad (A-4)$$

Using the Poisson density functions for  $n_1$  and  $n_2$ , we find that

$$P(E) \leq \sum_{n_2=0}^{\infty} \frac{[\gamma(B\tau)]^{n_2}}{n_2!} e^{-B\tau} \sum_{n_1=0}^{\infty} \frac{[(\frac{S+B}{\gamma}\tau)]^{n_1}}{n_1!} e^{-(S+B)\tau}$$

or equivalently that

$$P(E) \leq \exp \left\{ B\tau\gamma - B\tau + \frac{(S+B)\tau}{\gamma} - (S+B)\tau \right\}. \quad (A-5)$$

To make the bound as tight as possible, we select the  $\gamma$  which minimizes the exponent in (A-5). This minimizing  $\gamma$  satisfies the condition

$$B\tau - \frac{(S+B)\tau}{\gamma^2} = 0$$

or after dividing out common factors of  $\tau$ ,

$$\gamma = \sqrt{\frac{S+B}{B}}.$$

Substituting this minimizing  $\gamma$  into (A-5) and rearranging terms in the exponent, we find that

$$P(E) \leq \exp \left\{ -S\tau \left[ 1 - 2 \frac{B}{S} \left( \sqrt{1 + \frac{S}{B}} - 1 \right) \right] \right\} \quad (\text{A-6})$$

or that

$$P(E) \leq e^{-S\tau E_c(S/B)} \quad (\text{A-7})$$

where

$$E_c\left(\frac{S}{B}\right) = \left[ 1 - 2 \frac{B}{S} \left( \sqrt{1 + \frac{S}{B}} - 1 \right) \right] \quad (\text{A-8})$$

Rearranging terms in the expression for  $E_c(S/B)$ , we find that

$$E_c\left(\frac{S}{B}\right) = \frac{\left( \sqrt{1 + \frac{S}{B}} - 1 \right)^2}{\frac{S}{B}} \quad (\text{A-9})$$

Expressions (A-7) to (A-9) are identical to Eqs. (5) and (6) in the text.

APPENDIX B  
UPPER BOUND ON P(E) FOR OPTICAL DETECTION

This appendix derives Chernoff bound on  $P_M$  and  $P_F$ . The basic technique used to derive these bounds is the same as that in Appendix A. Thus, we will tend to skip lightly over the underlying arguments presented in Appendix A.

By definition

$$P_M = P[n \leq T/\lambda = S + B] \quad .$$

Let  $\alpha$  be a positive number greater than one. The positive quantity  $\alpha^{T-n}$  is greater than one whenever a miss occurs, and the expectation of  $\alpha^{T-n}$  given  $\lambda = S + B$  is an upper bound to  $P_M$ . Thus

$$P_M \leq \overline{\alpha^{T-n}} \quad .$$

This expectation may be calculated easily from the Poisson distribution.

$$\begin{aligned} \overline{\alpha^{T-n}} &= \alpha^T \sum_{n=0}^{\infty} \alpha^{-n} P(n/\lambda = S + B) \\ &= \alpha^T \sum_{n=0}^{\infty} \alpha^{-n} \frac{[(S + B) \tau]^n}{n!} e^{-(S+B)\tau} \\ &= \alpha^T e^{\frac{(S + B) \tau}{\alpha}} e^{-(S+B)\tau} \quad . \end{aligned}$$

Thus

$$P_M \leq e^{T \ln \alpha + (S+B)\tau \left[ \frac{1}{\alpha} - 1 \right]} \quad . \quad (B-1)$$

We now select the  $\alpha$  which gives the smallest upper bound on  $P_M$  by minimizing the exponent on the right side of Eq. (6) over  $\alpha$ . The minimizing  $\alpha$  is

$$\alpha = \frac{(S + B) \tau}{T} \quad . \quad (B-2)$$

Hence

$$P_M \leq e^{-S\tau \left[ \frac{B+S}{S} \right] \left[ \frac{\alpha - \ln \alpha - 1}{\alpha} \right]} \quad . \quad (B-3)$$

We may bound  $P_F$  in a similar way,

$$P_F = P[n \geq T/\lambda = B] \quad .$$

For  $\gamma \geq 1$ ,  $\gamma^{n-T}$  is always positive and greater than or equal to one whenever  $n \geq T$ . Thus

$$P_F \leq \overline{\gamma^{n-T}}$$



with the expectation taken with  $\lambda = B$ . Thus

$$P_F \leq \gamma^{-T} \sum_{n=0}^{\infty} \gamma^n \frac{(B\tau)^n}{n!} e^{-B\tau}$$

or equivalently,

$$P_F \leq e^{-T \ln \gamma + (B\tau)(\gamma-1)} \quad (B-4)$$

Obtaining the tightest upper bound by selecting the  $\gamma$  which minimizes the exponent, we select  $\gamma$  such that

$$\gamma = \frac{T}{B\tau} = \frac{T}{(S+B)\tau} \cdot \frac{S+B}{B} = \frac{S+B}{B} \cdot \frac{1}{\alpha} \quad (B-5)$$

Substituting (B-5) into (B-4) and using (14) and (B-2) for simplification, we find that

$$P_F \leq \frac{1}{\Gamma} e^{-S\tau \left[ \frac{B+S}{S} \right] \cdot \left[ \frac{\alpha - \ln \alpha - 1}{\alpha} \right]} \quad (B-6)$$

Thus (B-3) and (B-6) show that

$$P_M \leq e^{-S\tau \left[ 1 + \frac{B}{S} \right] \cdot \left[ \frac{\alpha - \ln \alpha - 1}{\alpha} \right]}$$

$$P_F \leq \frac{1}{\Gamma} e^{-S\tau \left[ 1 + \frac{B}{S} \right] \cdot \left[ \frac{\alpha - \ln \alpha - 1}{\alpha} \right]}$$

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